CLASSIFICATION OF SIMPLE W_n-MODULES WITH FINITE-DIMENSIONAL WEIGHT SPACES

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ABSTRACT. We classify all simple W_n-modules with finite-dimensional weight spaces. Every such module is either of a highest weight type or is a quotient of a module of tensor fields on a torus, which was conjectured by Eswara Rao. This generalizes the classical result of Mathieu on simple weight modules for the Virasoro algebra. In our proof of the classification we construct a functor from the category of cuspidal W_n-modules to the category of W_n-modules with a compatible action of the algebra of functions on a torus. We also present a new identity for certain quadratic elements in the universal enveloping algebra of W_1, which provides important information about cuspidal W_1-modules.

1. Introduction.

In 1992 Mathieu [20] classified simple modules with finite-dimensional weight spaces for the Virasoro algebra, proving a conjecture of Kac [14]. This was preceded by the work of Kaplansky [15] and Kaplansky-Santharoubane [16] on classification of simple modules for the Virasoro algebra with 1-dimensional weight spaces.

Mathieu proved that simple weight modules fall into two classes: (1) highest/lowest weight modules and (2) modules of tensor fields on a circle and their quotients. It was natural to ask the same question about the Lie algebra W_n of vector fields on a torus. In case of W_n, however, it was not clear at the time what should be the statement of the conjecture.

The W_n-modules of tensor fields were well-known, in 1970s Gelfand's school studied various cohomological problems related to these modules [11]. However it was not known whether there exist analogues of the highest weight modules for W_n with finite-dimensional weight spaces. It turned out that such modules do exist, and the highest weight type modules for W_n were constructed using a theorem of Berman-Billig [1], [5]. Recently vertex operator realizations of simple W_n-modules of the highest weight type were given in [4].

In 2004 Eswara Rao conjectured that simple modules for W_n with finite-dimensional weight spaces also fall into two classes: (1) modules of the highest weight type and (2) modules of tensor fields on a torus and their quotients.

Solenoidal Lie algebras (also known as centerless higher rank Virasoro algebras) serve as a bridge between W_1 and W_n. A solenoidal Lie algebra is the algebra of vector fields which are collinear at each point of the torus to some fixed generic vector. The name is motivated by the fact that the flow lines of such fields are dense windings on the torus. Solenoidal Lie algebras are very similar to W_1 in many regards, yet they have the same grading as W_n.

Su [24] classified simple cuspidal modules for the solenoidal Lie algebras (i.e., modules with a uniform bound on dimensions of weight spaces). Lu-Zhao [17]

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classified all simple modules with finite-dimensional weight spaces for this class of algebras.

Using this work, Mazorchuk-Zhao [21] described supports (sets of weights) of all simple W_n-modules with finite-dimensional weight spaces.

In the present paper we complete the classification of such modules, proving Eswara Rao's conjecture.

In order to achieve the result, we need to understand the structure of cuspidal (not necessarily simple) modules for W_1, the solenoidal algebras and W_n. For an important subcategory of cuspidal AW_n-modules that consists of modules admitting the action of the commutative algebra A of functions, this structure is understood, such AW_n-modules were analyzed in [9] and [2]. Our present approach reduces the classification of simple cuspidal W_n-modules to the classification of simple cuspidal AW_n-modules. To achieve this we construct the functor of coinduction, which associates to a W_n-module M a coinduced AW_n-module $\mathrm{Hom}(A, M)$. The coinduced module itself is too big and takes us out of the category of modules with finite-dimensional weight spaces. However, we can exploit the fact that W_n is itself an A-module and for each $x \in W_n$, $u \in M$ define $\psi(x, u) \in \mathrm{Hom}(A, M)$ by

$$\psi(x, u)(f) = (fx)u, \quad \text{for } f \in A.$$

A subspace in the coinduced module spanned by all $\psi(x, u)$ is an AW_n-submodule, which we call the A-cover $\widehat{M}$ of M. The A-cover construction plays a crucial role in our proof.

One of our key results is a theorem that the A-cover of a cuspidal module still has finite-dimensional weight spaces. As we have a homomorphism $\widehat{M} \to M$, $\psi(x, u) \mapsto xu$, which is surjective under a mild assumption $W_n M = M$, we are able to reduce the classification of simple cuspidal W_n-modules to the classification of simple cuspidal AW_n-modules, obtained in [9].

The proof of the theorem that the A-cover of a cuspidal module is again cuspidal, is based on an interesting new identity for the following quadratic elements in the universal enveloping algebra of W_1:

$$\Omega^{(m)}_{k,s} = \sum_{a=0}^{m} (-1)^a \binom{m}{a} e_{k-a} e_{s+a}.$$

We prove, in particular, that for $m \geq 2$

$$\sum_{i=0}^{m}\sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i}\binom{m}{j} \left(\left\{ \Omega^{(m)}_{k-i,s-j}, \Omega^{(m)}_{q+i,p+j} \right\} - \left\{ \Omega^{(m)}_{k-i,q-j}, \Omega^{(m)}_{s+i,p+j} \right\} \right)$$
$$= (q-s)(p-k+2m)\Omega^{(4m)}_{k+p+2m,s+q-2m}, \tag{1.1}$$

where by $\{X, Y\}$ we denote the anticommutator $XY + YX$.

Using this algebraic identity we show that operators $\Omega^{(m)}_{k,s}$ annihilate any cuspidal W_1-module M for large enough m, which depends only on the length of the composition series of M.

Let us present the classification of simple W_n-modules with finite-dimensional weight spaces that we establish in this paper.

Modules of tensor fields on a torus $T(U, \beta)$ are parametrized by their support $\beta + \mathbb{Z}^n$ and a finite-dimensional gl_n-module U. The latter is explained by the fact that the local coordinate changes induce the action of gl_n on a fiber of a tensor

bundle. If a gl_n-module U is simple the resulting module $T(U,\beta)$ is a simple W_n-module, except when it is a module $\Omega^k(\beta)$ of differential k-forms. The exceptional nature of the modules of differential forms comes from the fact that they form the de Rham complex, and the differential d is a homomorphism of W_n-modules.

Modules of the highest weight type are constructed using the technique of the generalized Verma modules. We choose a $\mathbb{Z}$-grading on W_n, say by degree in t_n. Then a degree zero subalgebra is a semidirect product of W_{n-1} with an abelian ideal. We take a tensor module for W_{n-1} and let the abelian ideal act on it via A-action rescaled with a complex parameter γ, yielding a module $T(U,\beta,\gamma)$ for the degree zero subalgebra in W_n. We postulate that the positive degree subalgebra acts trivially on $T(U,\beta,\gamma)$ and construct the induced W_n-module $M(U,\beta,\gamma)$. By Theorem of Berman-Billig [1], the simple quotient $L(U,\beta,\gamma)$ has finite-dimensional weight spaces. Finally, we can twist such modules by a $GL_n(\mathbb{Z})$-automorphism of W_n, which corresponds to a change in the $\mathbb{Z}$-grading on W_n.

The trivial 1-dimensional W_n-module can be constructed as either a quotient of a module of differential forms or as a module of the highest weight type.

Our main result is the following theorem.

Theorem 1.1. *(cf., Theorem 5.4) Every simple weight W_n-module with finite-dimensional weight spaces is isomorphic to one of the following:*

- *a module of tensor fields $T(U,\beta)$, where $\beta \in \mathbb{C}^n$ and U is a simple finite-dimensional gl_n-module, different from an exterior power of the natural n-dimensional gl_n-module,*
- *a submodule $d\Omega^k(\beta) \subset \Omega^{k+1}(\beta)$ for $0 \le k < n$ and $\beta \in \mathbb{C}^n$,*
- *a module of the highest weight type $L(U,\beta,\gamma)^g$, twisted by $g \in GL_n(\mathbb{Z})$, where U is a simple finite-dimensional gl_{n-1}-module, $\beta \in \mathbb{C}^{n-1}$ and $\gamma \in \mathbb{C}$.*

The structure of the paper is as follows. In Section 2 we discuss Lie algebras of vector fields on a torus, their solenoidal subalgebras and give basic definitions in their representation theory. In Section 3 we prove our key identity (1.1) and link it with annihilators of cuspidal modules. In Section 4 we develop the theory of coinduced modules and study the properties of A-covers of cuspidal modules. Finally, in Section 5 we apply our results to establish the classification of simple W_n-modules with finite-dimensional weight spaces.

2. Lie algebras of vector fields and their cuspidal modules

Consider the Lie algebra W_n of vector fields on an n-dimensional torus $\mathbb{T}^n$. The algebra of (complex-valued) Fourier polynomials on $\mathbb{T}^n$ is isomorphic to the algebra of Laurent polynomials

$$A = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}],$$

and W_n is the Lie algebra of derivations of A. Thus W_n has a natural structure of an A-module, which is free of rank n. We choose $d_1 = t_1 \frac{\partial}{\partial t_1}, \dots, d_n = t_n \frac{\partial}{\partial t_n}$ as a basis of this A-module:

$$W_n = \mathop{\oplus}_{p=1}^{n} A d_p.$$

Setting the notation $t^r = t_1^{r_1} \dots t_n^{r_n}$ for $r = (r_1, \dots, r_n) \in \mathbb{Z}^n$, we can write the Lie bracket in W_n as follows:

$$[t^r d_a, t^s d_b] = s_a t^{r+s} d_b - r_b t^{r+s} d_a, \quad a, b = 1, \dots, n; \; r, s \in \mathbb{Z}^n.$$

The subspace spanned by $d_1, \dots, d_n$ is the Cartan subalgebra in W_n and the adjoint action of these elements induces a $\mathbb{Z}^n$-grading.

In case when $n = 1$ we get the Witt algebra W_1 with the basis $e_k = t_1^k d_1$, $k \in \mathbb{Z}$, and the bracket

$$[e_r, e_s] = (s - r)e_{r+s}. \tag{2.1}$$

One particular family of subalgebras in W_n will play a significant role in this paper.

Definition 2.1. We call a vector $\mu \in \mathbb{C}^n$ *generic* if $\mu \cdot r \neq 0$ for all $r \in \mathbb{Z}^n \backslash \{0\}$.

Definition 2.2. Let μ be a generic vector in $\mathbb{C}^n$ and let $d_\mu = \mu_1 d_1 + \ldots + \mu_n d_n$. A *solenoidal* Lie algebra W_μ is the subalgebra in W_n which consists of vector fields collinear to μ at each point of $\mathbb{T}^n$, $W_\mu = A d_\mu$.

The Lie bracket in W_μ is

$$[t^r d_\mu, t^s d_\mu] = \mu \cdot (s - r) t^{r+s} d_\mu.$$

Denote by Γ_μ the image of $\mathbb{Z}^n$ under the embedding $\mathbb{Z}^n \to \mathbb{C}$ given by $r \mapsto \mu \cdot r$. Then we can view the solenoidal Lie algebra W_μ as a version of the Witt algebra W_1 where the indices of the basis elements e_r run not over $\mathbb{Z}$, but over the lattice $\Gamma_\mu \subset \mathbb{C}$. Here we make the identification $t^r d_\mu = e_{\mu \cdot r}$ and the formula (2.1) remains valid.

The properties of the solenoidal Lie algebras are very similar to those of W_1, yet these algebras are close enough to W_n. In particular, W_n can be decomposed (as a vector space) in a direct sum of n solenoidal subalgebras. This elementary observation will be important for our proof.

Let us now discuss modules for the Lie algebras of vector fields. A W_n-module M is called a *weight* module if $M = \bigoplus_{\lambda \in \mathbb{C}^n} M_\lambda$, where the weight space M_λ is defined as

$$M_\lambda = \{ v \in M \,|\, d_i v = \lambda_i v \quad \text{for} \quad i = 1, \dots, n \}.$$

In particular, W_n is a weight module over itself and its weight decomposition coincides with its $\mathbb{Z}^n$-grading.

Weight spaces of a W_μ-module are defined as the eigenspaces of d_μ.

Any weight W_n-module can be decomposed into a direct sum of submodules corresponding to distinct cosets of $\mathbb{Z}^n$ in $\mathbb{C}^n$. It is then sufficient to study modules supported on a single coset $\beta + \mathbb{Z}^n$, $\beta \in \mathbb{C}^n$. We are going to impose such an assumption on M for the rest of this paper.

Definition 2.3. We call M an AW_n-module if it is a module for both the Lie algebra W_n and the commutative unital algebra A of functions on a torus, with these two structures being compatible:

$$x(fv) = (xf)v + f(xv), \quad f \in A, x \in W_n, v \in M. \tag{2.2}$$

For solenoidal Lie algebras the concept of an AW_μ-module is defined in the same way.

If an AW_n-module M is a weight module then (2.2) implies that the action of A is compatible with the weight grading of M: $A_\gamma M_\lambda \subset M_{\lambda+\gamma}$, $\gamma \in \mathbb{Z}^n$, $\lambda \in \beta + \mathbb{Z}^n$. Suppose an AW_n-module M has a weight decomposition with one of the weight spaces being finite-dimensional. Since all non-zero homogeneous elements of A are invertible, we conclude that all weight spaces of M have the same dimension and that M is a free A-module of a finite rank.

Definition 2.4. A weight module is called *cuspidal* if the dimensions of its weight spaces are uniformly bounded by some constant.

Irreducible and indecomposable AW_n-modules with finite-dimensional weight spaces are classified in [9] and [2]. Our strategy will be to use this classification in order to describe simple cuspidal W_n-modules.

For the Witt Lie algebra W_1 the classification of simple weight modules was obtained by Mathieu [20]. Every such module is either a highest/lowest weight module, a module of tensor fields, or a quotient of the module of functions on a circle. Apart from the trivial 1-dimensional module, highest/lowest weight W_1-modules are not cuspidal ([18], Corollary III.3), while the modules of tensor fields are cuspidal and have weight spaces of dimension at most 1.

Let us define W_1-modules of tensor fields on a circle. For $\alpha, \beta \in \mathbb{C}$ let $T(\alpha, \beta)$ be a vector space with a basis $\{v_s \,|\, s \in \beta + \mathbb{Z}\}$ and the following action of W_1:

$$e_k v_s = (s + \alpha k) v_{s+k}. \tag{2.3}$$

The modules $T(\alpha, \beta)$ are simple unless $\alpha = 0, 1$ and $\beta + \mathbb{Z} = \mathbb{Z}$. The exceptional cases $\alpha = 0, 1$ are the modules of functions and 1-forms on a circle. The module of functions $T(0,0)$ has a 1-dimensional submodule (constant functions), and the quotient by this 1-dimensional submodule is a simple W_1-module $\overline{T}(0,0)$ with a "hole" in a zero weight space. Applying the differential map from functions to 1-forms we get that $\overline{T}(0,0)$ is a submodule in $T(1,0)$, with the quotient of $T(1,0)$ by $\overline{T}(0,0)$ being a 1-dimensional module.

Simple cuspidal modules for the solenoidal Lie algebra W_μ have the same description as in the case of W_1 [24], with the modification that the basis vectors v_s are indexed by $s \in \beta + \Gamma_\mu$, and the action of W_μ is still given by the formula (2.3) with $k \in \Gamma_\mu$. The W_μ-modules $T(\alpha, \beta)$ are simple unless $\alpha = 0, 1$ and $\beta + \Gamma_\mu = \Gamma_\mu$.

3. Annihilators of cuspidal modules

In this section we are going to establish an identity in the universal enveloping algebra of W_1 which will be instrumental for the classification of simple cuspidal W_n-modules. Using this important identity we will be able to show that certain quadratic elements of the universal enveloping algebra belong to annihilators of cuspidal modules for W_1 and W_μ. Let us mention that annihilators of simple W_1-modules $T(\alpha, \beta)$ were described in [7]. Our results also shed light on the structure of extensions of cuspidal W_1-modules. Such extensions of length 2 and 3 were studied in [19] and [6].

Let us inductively define *differentiators* $\Omega^{(m)}_{k,s} \in U(W_1)$. Set

$$\Omega^{(0)}_{k,s} = e_k e_s$$

and let

$$\Omega^{(m+1)}_{k,s} = \Omega^{(m)}_{k,s} - \Omega^{(m)}_{k-1,s+1}.$$

The element $\Omega^{(m)}_{k,s}$ can be viewed as the m-th difference derivative of $\Omega^{(0)}_{k,s}$. We can express the differentiators in a closed form:

$$\Omega^{(m)}_{k,s} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} e_{k-i} e_{s+i}.$$

Example 3.1. $\Omega^{(2)}_{1,-1} = e_1 e_{-1} - 2e_0^2 + e_{-1}e_1$ is the Casimir element of $sl_2 = \langle e_{-1}, e_0, e_1\rangle$.

The following Lemma can be easily established by a direct computation:

Lemma 3.2. *For all $k, s \in \mathbb{Z}$, the differentiators $\Omega^{(3)}_{k,s}$ belong to the annihilators of all W_1-modules of tensor fields $T(\alpha,\beta)$.*

More generally, if M is a module with a polynomial action of W_1 (see [1] for the exact definition), then $\Omega^{(m)}_{k,s} \in \mathrm{Ann}(M)$ for large enough m.

Now let us state our key identity. We denote by $\{X, Y\}$ the anticommutator $XY + YX$.

Theorem 3.3. *Let $m, r \geq 2$. Then*

$$\sum_{i=0}^{m}\sum_{j=0}^{r}(-1)^{i+j}\binom{m}{i}\binom{r}{j}\left(\left\{\Omega^{(m)}_{k-i,s-j}, \Omega^{(r)}_{q+i,p+j}\right\} - \left\{\Omega^{(m)}_{k-i,q-j}, \Omega^{(r)}_{s+i,p+j}\right\}\right)$$
$$= (q-s)\left((p-k+2r)\Omega^{(2m+2r-1)}_{k+p+2r,s+q-2r} - (p-k+2m)\Omega^{(2m+2r-1)}_{k+p+2r-1,s+q-2r+1}\right).$$

Corollary 3.4. *For every ℓ there exists m such that for all k, s the differentiator $\Omega^{(m)}_{k,s}$ annihilates every cuspidal W_1-module with a composition series of length ℓ.*

Proof of Corollary 3.4. Let M be a cuspidal W_1-module with a composition series of length ℓ:

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_\ell = M$$

with simple quotients M_{i+1}/M_i. We give a proof by induction on ℓ. Since non-trivial highest/lowest weight W_1-modules are not cuspidal, the composition series of M will have the modules $T(\alpha,\beta)$, $\overline{T}(0,0)$ and the trivial 1-dimensional module as possible simple quotients . By Lemma 3.2, $\Omega^{(3)}_{k,s}$ belongs to the annihilator of every cuspidal module of length $\ell = 1$. To prove the step of induction, assume $\ell > 1$ and write $\ell = \ell_1 + \ell_2$ with $\ell_1, \ell_2 \geq 1$. By induction assumption there exist m, r such that $\Omega^{(m)}_{k,s}$, $\Omega^{(r)}_{p,q}$ annihilate all cuspidal modules of lengths ℓ_1, ℓ_2 respectively. Then both $\Omega^{(m)}_{k,s}\Omega^{(r)}_{p,q}$ and $\Omega^{(r)}_{p,q}\Omega^{(m)}_{k,s}$ annihilate M. Indeed, $\Omega^{(r)}_{p,q}M \subset M_{\ell_1}$ since $\Omega^{(r)}_{p,q} \in \mathrm{Ann}\,(M/M_{\ell_1})$ and thus $\Omega^{(m)}_{k,s}\Omega^{(r)}_{p,q}M = 0$. The argument for $\Omega^{(r)}_{p,q}\Omega^{(m)}_{k,s}$ is analogous.

If $m = r$ then Theorem 3.3 implies that $\Omega^{(4m)}_{k,s} \in \mathrm{Ann}(M)$. If $m \neq r$ we get that for all p, k, s, q the expression

$$(p-k+2r)\Omega^{(2m+2r-1)}_{k+p+2r,s+q-2r} - (p-k+2m)\Omega^{(2m+2r-1)}_{k+p+2r-1,s+q-2r+1}$$

annihilates M. However we can vary the value of $p-k$ while keeping $p+k$ fixed. This implies that $\Omega^{(2m+2r-1)}_{k+p+2r,s+q-2r} \in \mathrm{Ann}(M)$, which completes the proof of the corollary. $\square$

Example 3.5. Our result implies that $\Omega^{(12)}_{k,s}$ annihilates all cuspidal W_1-modules with a composition series of length 2. Such modules were classified by Martin-Piard in [19]. The answer turns out to be essentially the same as in the case of Lie algebra

of vector fields on a line, which was investigated earlier by Feigin-Fuks [10]. Feigin-Fuks listed explicit formulas for such extensions, which are also valid for W_1. Here is an extension M from their list:

$$0 \to T\left(\frac{7-\sqrt{19}}{2}, \beta\right) \to M \to T\left(\frac{-5-\sqrt{19}}{2}, \beta\right) \to 0$$

with basis $\{u_p, w_p \mid p \in \beta + \mathbb{Z}\}$ and the W_1-action:

$$\begin{aligned}
e_k u_p &= \left(p + \frac{7-\sqrt{19}}{2}k\right) u_{p+k}, \\
e_k w_p &= \left(p + \frac{-5-\sqrt{19}}{2}k\right) w_{p+k} \\
&+ \left(-\frac{22+5\sqrt{19}}{4}k^7 - \frac{31+7\sqrt{19}}{2}k^6 p - \frac{25+7\sqrt{19}}{2}k^5 p^2 - 5k^4p^3 + 5k^3p^4 + 2k^2p^5\right) u_{p+k}.
\end{aligned}$$

We can see that $\Omega^{(0)}_{k,s} w_p = e_k e_s w_p$ has coefficients which are polynomials of degree 8 in k, s. Thus $\Omega^{(9)}_{k,s} w_p = 0$. By inspecting the table of possible extensions given in [10] one can see that the lowest value of m for which $\Omega^{(m)}_{k,s}$ annihilates all cuspidal modules of length $\ell = 2$ is $m = 9$.

Proof of Theorem 3.3.

$$\begin{aligned}
&\sum_{i=0}^{m}\sum_{j=0}^{r}(-1)^{i+j}\binom{m}{i}\binom{r}{j}\left(\left\{\Omega^{(m)}_{k-i,s-j}, \Omega^{(r)}_{q+i,p+j}\right\} - \left\{\Omega^{(m)}_{k-i,q-j}, \Omega^{(r)}_{s+i,p+j}\right\}\right) \\
&\quad = \sum_{i=0}^{m}\sum_{j=0}^{r}\sum_{a=0}^{m}\sum_{b=0}^{r}(-1)^{i+j+a+b}\binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} \\
&\qquad\qquad \times\Big(e_{k-i-a}e_{s-j+a}e_{q+i-b}e_{p+j+b} + e_{q+i-b}e_{p+j+b}e_{k-i-a}e_{s-j+a} \\
&\qquad\qquad\quad - e_{k-i-a}e_{q-j+a}e_{s+i-b}e_{p+j+b} - e_{s+i-b}e_{p+j+b}e_{k-i-a}e_{q-j+a}\Big).
\end{aligned}$$

Let us switch indices a with i and b with j in the last two summands. We get

$$\begin{aligned}
&\sum_{a,i=0}^{m}\sum_{b,j=0}^{r}(-1)^{i+j+a+b}\binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} \\
&\qquad \times\Big(e_{k-i-a}e_{s-j+a}e_{q+i-b}e_{p+j+b} + e_{q+i-b}e_{p+j+b}e_{k-i-a}e_{s-j+a} \\
&\qquad\quad - e_{k-i-a}e_{q+i-b}e_{s-j+a}e_{p+j+b} - e_{s-j+a}e_{p+j+b}e_{k-i-a}e_{q+i-b}\Big)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{a,i=0}^{m} \sum_{b,j=0}^{r} (-1)^{i+j+a+b} \binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} \\
&\times \Big(e_{k-i-a}[e_{s-j+a}, e_{q+i-b}]e_{p+j+b} + e_{p+j+b}e_{k-i-a}[e_{q+i-b}, e_{s-j+a}] \\
&+ [e_{q+i-b}, e_{p+j+b}]e_{k-i-a}e_{s-j+a} + e_{p+j+b}[e_{q+i-b}, e_{k-i-a}]e_{s-j+a} \\
&+ e_{p+j+b}[e_{k-i-a}, e_{s-j+a}]e_{q+i-b} + [e_{p+j+b}, e_{s-j+a}]e_{k-i-a}e_{q+i-b} \Big).
\end{aligned}$$

Here the part involving the last 4 summands vanishes. The argument in each case is the same, so we will show this for the last term, which equals:

$$\sum_{a,i=0}^{m} \sum_{b,j=0}^{r} (-1)^{i+j+a+b} \binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} (s-p-2j+a-b)e_{s+p+a+b}e_{k-i-a}e_{q+i-b}.$$

We can see that the resulting monomials are independent of j. Taking the sum in j we get zero, since for $r \geq 2$

$$\sum_{j=0}^{r}(-1)^j\binom{r}{j} = \sum_{j=0}^{r}(-1)^j j\binom{r}{j} = 0.$$

The sum with the first 2 summands may be expressed in the following way:

$$\begin{aligned}
&\sum_{a,i=0}^{m} \sum_{b,j=0}^{r} (-1)^{i+j+a+b} \binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} \\
&\times \Big([e_{k-i-a}, e_{p+j+b}][e_{s-j+a}, e_{q+i-b}] + e_{k-i-a}[[e_{s-j+a}, e_{q+i-b}], e_{p+j+b}] \Big).
\end{aligned}$$

The sum involving the last term again vanishes since the monomial in the resulting expression

$$(q-s+i+j-a-b)(p-s-q-i+2j-a+2b)e_{k-i-a}e_{p+s+q+i+a}$$

is independent of j and b and evaluating the sum in both j and b we get zero since every term in the expansion of the coefficient vanishes under summation in one of these two indices. Thus we are left with

$$\begin{aligned}
&\sum_{a,i=0}^{m} \sum_{b,j=0}^{r} (-1)^{i+j+a+b} \binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} \\
&\qquad \times (q-s+i+j-a-b)(p-k+i+j+a+b)e_{k+p-i+j-a+b}e_{s+q+i-j+a-b}.
\end{aligned}$$

In the above expression everything except the factor $q-s+i+j-a-b$ is symmetric in $\{i,a\}$ and $\{j,b\}$. We can symmetrize in these pairs of variables and get

$$\begin{aligned}
&(q-s) \sum_{a,i=0}^{m} \sum_{b,j=0}^{r} (-1)^{i+j+a+b} \binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} \\
&\qquad\qquad \times (p-k+(i+a)+(j+b))e_{k+p-i+j-a+b}e_{s+q+i-j+a-b}.
\end{aligned}$$

Make a change of variables replacing j with $r-j$ and b with $r-b$:

$$\begin{aligned}
&(q-s) \sum_{a,i=0}^{m} \sum_{b,j=0}^{r} (-1)^{i+j+a+b} \binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j} \\
&\qquad\qquad \times (p-k+2r+(i+a)-(j+b))e_{k+p+2r-i-j-a-b}e_{s+q-2r+i+j+a+b}.
\end{aligned}$$

Set $u = i + j + a + b$, which ranges from 0 to $2m + 2r$. Then the coefficient at $e_{k+p+2r-u}e_{s+q-2r+u}$ is the same as the coefficient at t^u in

$$\begin{aligned}
&(q-s)\sum_{a,i=0}^{m}\sum_{b,j=0}^{r}(-1)^{i+j+a+b}\binom{m}{a}\binom{r}{b}\binom{m}{i}\binom{r}{j}(p-k+2r+(i+a)-(j+b))t^{i+j+a+b}\\
&\quad=(q-s)\big((p-k+2r)(1-t)^m(1-t)^m(1-t)^r(1-t)^r\\
&\qquad+2(1-t)^{2r+m}t\frac{d}{dt}(1-t)^m-2(1-t)^{2m+r}t\frac{d}{dt}(1-t)^r\big)\\
&\quad=(q-s)\left((p-k+2r)(1-t)^{2m+2r}-2mt(1-t)^{2m+2r-1}+2rt(1-t)^{2m+2r-1}\right)\\
&\quad=(q-s)\left((p-k+2r)(1-t)^{2m+2r-1}-(p-k+2m)t(1-t)^{2m+2r-1}\right).
\end{aligned}$$

The coefficient at t^u is

$$(q-s)\left((p-k+2r)(-1)^u\binom{2m+2r-1}{u}-(p-k+2m)(-1)^{u-1}\binom{2m+2r-1}{u-1}\right),$$

which is the same as the coefficient at $e_{k+p+2r-u}e_{s+q-2r+u}$ in

$$(q-s)\left((p-k+2r)\Omega^{(2m+2r-1)}_{k+p+2r,s+q-2r}-(p-k+2m)\Omega^{(2m+2r-1)}_{k+p+2r-1,s+q-2r+1}\right),$$

which completes the proof of the Theorem. □

Remark 3.6. In the proofs of Theorem 3.3 and Corollary 3.4 we relied little on the specific expressions for the structure constants for the Lie bracket in W_1 and its action on simple cuspidal modules. What matters is the fact that the structure constants are polynomial. Thus our methods are applicable to a wider class of Lie algebras with polynomial multiplication and their modules with polynomial action, which were introduced in [1].

Remark 3.7. We can see from Theorem 3.3 that the subspace in $U(W_1)$ spanned by the anticommutators of differentiators contains differentiators of higher order. Rather surprisingly, it appears that the subspace spanned by the commutators of the differentiators does not contain any non-zero quadratic elements.

Let us now state the analogue of Theorem 3.3 for the solenoidal Lie algebra W_μ. For $k, s, h \in \Gamma_\mu$, $m \geq 0$, define the differentiators in $U(W_\mu)$:

$$\Omega^{(m,h)}_{k,s} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}e_{k-ih}e_{s+ih}.$$

We have the following

Theorem 3.8. *Let $m, r \geq 2$, $k, s, p, q, h \in \Gamma_\mu \subset \mathbb{C}$. Then*

$$\begin{aligned}
&\sum_{i=0}^{m}\sum_{j=0}^{r}(-1)^{i+j}\binom{m}{i}\binom{r}{j}\left(\left\{\Omega^{(m,h)}_{k-ih,s-jh},\Omega^{(r,h)}_{q+ih,p+jh}\right\}-\left\{\Omega^{(m,h)}_{k-ih,q-jh},\Omega^{(r,h)}_{s+ih,p+jh}\right\}\right)\\
&=(q-s)\left((p-k+2rh)\Omega^{(2m+2r-1,h)}_{k+p+2rh,s+q-2rh}-(p-k+2mh)\Omega^{(2m+2r-1,h)}_{k+p+(2r-1)h,s+q-(2r-1)h}\right).
\end{aligned}$$

Corollary 3.9. *For every ℓ there exists m such that for all $k, s, h \in \Gamma_\mu$ the differentiator $\Omega^{(m,h)}_{k,s}$ annihilates every cuspidal W_μ-module with a composition series of length ℓ.*

The proofs of Theorem 3.8 and Corollary 3.9 are exactly the same as in the case of W_1.

4. Coinduced modules

In this section $\mathcal{L}$ will denote either W_n or W_μ. The goal of this section is to prove the following

Theorem 4.1. *Let M be a cuspidal $\mathcal{L}$-module satisfying $\mathcal{L}M = M$. Then there exist a cuspidal $A\mathcal{L}$-module $\widehat{M}$ and a surjective homomorphism of $\mathcal{L}$-modules:*

$$\widehat{M} \to M.$$

Our main tool will be the coinduction functor. Let M be an $\mathcal{L}$-module.

Definition 4.2. A module *coinduced* from M is the space $\mathrm{Hom}(A, M)$ with the following action of $\mathcal{L}$ and A:

$$(x\phi)(f) = x(\phi(f)) - \phi(x(f)), \tag{4.1}$$

$$(g\phi)(f) = \phi(gf), \quad \text{for } \phi \in \mathrm{Hom}(A, M),\ x \in \mathcal{L},\ f, g \in A. \tag{4.2}$$

Proposition 4.3. *(1) The coinduced module* $\mathrm{Hom}(A, M)$ *is an $A\mathcal{L}$-module.*
(2) The map

$$\pi : \mathrm{Hom}(A, M) \to M,$$
$$\phi \mapsto \phi(1),$$

is a surjective homomorphism of $\mathcal{L}$-modules.

Proof. Let us first verify that (4.1) defines an $\mathcal{L}$-module structure on $\mathrm{Hom}(A, M)$:

$$\begin{aligned}
&(x(y\phi))(f) - (y(x\phi))(f) = x((y\phi)(f)) - (y\phi)(xf) - y((x\phi)(f)) + (x\phi)(yf) \\
&= x(y(\phi(f))) - x(\phi(yf)) - y(\phi(xf)) + \phi(y(xf)) \\
&- y(x(\phi(f))) + y(\phi(xf)) + x(\phi(yf)) - \phi(x(yf)) \\
&= [x, y](\phi(f)) - \phi([x, y]f) = ([x, y]\phi)(f), \quad \text{for } \phi \in \mathrm{Hom}(A, M),\ x, y \in \mathcal{L},\ f \in A.
\end{aligned}$$

Clearly, (4.2) defines an A-module structure on $\mathrm{Hom}(A, M)$. To show that the $\mathcal{L}$-module and the A-module structures are compatible, we need to check that

$$x(g\phi) = x(g)\phi + g(x\phi).$$

Indeed,

$$\begin{aligned}
&(x(g)\phi)(f) + (gx(\phi))(f) = \phi(x(g)f) + x(\phi)(gf) = \phi(x(g)f) + x(\phi(gf)) - \phi(x(gf)) \\
&= \phi(x(g)f) + x(\phi(gf)) - \phi(x(g)f) - \phi(gx(f)) = x((g\phi)f) - (g\phi)(xf) = (x(g\phi))(f).
\end{aligned}$$

In order to establish the claim of part (2), we need to show that $x\phi$ is mapped to $x(\phi(1))$, i.e., $(x\phi)(1) = x(\phi(1))$, which is true since $x(1) = 0$. The surjectivity of π is obvious. □

The shortcoming of the coinduced module $\mathrm{Hom}(A, M)$ is that it is too big. We are now going to construct a smaller submodule in $\mathrm{Hom}(A, M)$ which will serve well for our purposes.

Definition 4.4. An A-*cover* of an $\mathcal{L}$-module M is the subspace $\widehat{M} \subset \mathrm{Hom}(A, M)$ spanned by the set $\{\psi(x,u) \mid x \in \mathcal{L}, u \in M\}$, where $\psi(x,u) \in \mathrm{Hom}(A, M)$ is given by

$$\psi(x,u)(f) = (fx)u, \quad \text{for } f \in A.$$

Proposition 4.5. *(1) The action of $\mathcal{L}$ and A on $\widehat{M}$ is the following:*

$$y\psi(x,u) = \psi([y,x],u) + \psi(x,yu), \tag{4.3}$$

$$g\psi(x,u) = \psi(gx,u), \quad \text{for } x,y \in \mathcal{L},\ u \in M,\ g \in A. \tag{4.4}$$

(2) If M is a weight module then so is $\widehat{M}$.

(3) $\pi(\widehat{M}) = \mathcal{L}M$.

Proof. In proving (1) we are going to use the fact that the adjoint representation of $\mathcal{L}$ is an $A\mathcal{L}$-module:

$$[y, fx] = y(f)x + f[y,x].$$

Then

$$\begin{aligned}&(y\psi(x,u))(f) = y(\psi(x,u)(f)) - \psi(x,u)(yf) = y((fx)u) - ((yf)x)u \\ &= [y,fx]u + (fx)(yu) - ((yf)x)u = ((yf)x)u + (f[y,x])u + (fx)(yu) - ((yf)x)u \\ &= \psi([y,x],u)(f) + \psi(x,yu)(f).\end{aligned}$$

For the action of A we have

$$(g\psi(x,u))(f) = \psi(x,u)(gf) = ((fg)x)u = \psi(gx,u)(f).$$

Let us prove the second part of the proposition. If M is a weight module then $\widehat{M}$ is spanned by $\psi(x,u)$ with x and u being eigenvectors with respect to the action of the Cartan subalgebra of $\mathcal{L}$. Then (4.3) shows that $\psi(x,u)$ is also an eigenvector for the action of the Cartan subalgebra. Since $\widehat{M}$ is spanned by its weight vectors, it is a weight module.

Finally, for part (3) we note that $\pi(\psi(x,u)) = \psi(x,u)(1) = xu$, which implies the claim. $\square$

Remark 4.6. It follows from (4.3) that the A-cover $\widehat{M}$ can also be constructed as a quotient of $\mathcal{L} \otimes M$ (where the first tensor factor is the adjoint module) by a submodule

$$\left\{ \sum_i x_i \otimes u_i \;\middle|\; \sum_i (fx_i)u_i = 0 \quad \text{for all } f \in A \right\}.$$

Example 4.7. One of the technical difficulties in working with W_n-modules is the existence of modules with "holes" and "bumps" at the zero weight space. Let us show how the use of A-cover remedies this complication. Consider a W_1-module $M = \overline{T}(0,0) = \mathbb{C}[t,t^{-1}]/\langle 1 \rangle$. We fix a spanning set $u_j = t^j$ with a provision $u_0 = 0$. The action of W_1 is $e_k u_s = s u_{s+k}$. Let us construct the A-cover of M:

$$\psi(e_k, u_s)(t^m) = (t^m e_k)u_s = e_{k+m}u_s = s u_{s+k+m}.$$

Introduce $\theta_j \in \mathrm{Hom}(A, M)$, given by

$$\theta_j(t^m) = u_{m+j}.$$

Then we can see that $\psi(e_k, u_s) = s\theta_{k+s}$. Thus $\widehat{M}$ has a basis $\{\theta_j | j \in \mathbb{Z}\}$ with the AW_1-action $e_p\theta_j = j\theta_{j+p}$, $t^p\theta_j = \theta_{j+p}$. Note that $\theta_0 \neq 0$ and $\widehat{M} \cong \mathbb{C}[t,t^{-1}]$. We see that taking the A-cover fills in the hole in M.

Example 4.8. Let M be the adjoint representation for the Virasoro algebra, viewed as a W_1-module. The W_1-action can be written as

$$\begin{aligned} e_k u_j &= (j-k)u_{j+k} + \delta_{k+j,0} k^3 z, \\ e_k z &= 0. \end{aligned}$$

Then $\psi(e_k, u_j)(t^m) = e_{k+m} u_j = (j-k-m)u_{j+k+m} + \delta_{k+m+j,0}(-j)^3 z$, while $\psi(e_k, z) = 0$. Consider the following elements in $\mathrm{Hom}(A, M)$:

$$\begin{aligned} \tau_j(t^m) &= (j+m)u_{j+m}, \\ \theta_j(t^m) &= u_{j+m}, \\ \eta_j(t^m) &= \delta_{j+m,0} z. \end{aligned}$$

Then $\psi(e_k, u_j) = -\tau_{k+j} + 2j\theta_{k+j} - j^3 \eta_{k+j}$. One can check that the A-action is $t^p \tau_j = \tau_{j+p}$, $t^p \theta_j = \theta_{j+p}$, $t^p \eta_j = \eta_{j+p}$, while W_1 acts in the following way:

$$\begin{aligned} e_p \tau_j &= (j-2p)\tau_{j+p} + 2p^2 \theta_{j+p} - p^4 \eta_{j+p}, \\ e_p \theta_j &= (j-p)\theta_{j+p} + p^3 \eta_{j+p}, \\ e_p \eta_j &= (j+p)\eta_{j+p}. \end{aligned}$$

In this case $\widehat{M}$ is a weight module with 3-dimensional weight spaces. The map $\pi : \widehat{M} \to M$ is given by $\pi(\tau_j) = j u_j$, $\pi(\theta_j) = u_j$, $\pi(\eta_j) = \delta_{j,0} z$.

Assume now that M is a cuspidal $\mathcal{L}$-module. Its A-cover $\widehat{M}$ is a weight module, however it is not a priori clear that the weight spaces of $\widehat{M}$ are finite-dimensional. In the above examples this was the case due to the fact that the structure constants in the modules were polynomial. We are going to prove next that $\widehat{M}$ is indeed cuspidal. The key ingredient in the proof is Corollary 3.9. We begin with the solenoidal Lie algebra.

Theorem 4.9. *Let M be a cuspidal module for the solenoidal Lie algebra W_μ. Then the A-cover $\widehat{M}$ of M is cuspidal.*

Proof. Without loss of generality let us assume that $\beta = 0$ in case when $\beta + \Gamma_\mu = \Gamma_\mu$. This ensures that e_0 acts with a non-zero scalar on every weight space of M different from M_β.

For $k \in \Gamma_\mu$, $\lambda \in \beta + \Gamma_\mu$ denote by $\psi(e_k, M_\lambda)$ a subspace

$$\psi(e_k, M_\lambda) = \left\{ \psi(e_k, u) \;\middle|\; u \in M_\lambda \right\} \subset \widehat{M}.$$

Since weight spaces of M are finite dimensional, so are the spaces $\psi(e_k, M_\lambda)$.

Since $\widehat{M}$ is an A-module, it is sufficient to show that one of its weight spaces is finite-dimensional, then all other weight spaces will have the same dimension. Fix a weight $\beta + s$, $s \in \Gamma_\mu$ and let us prove that $\widehat{M}_{\beta+s}$ is finite-dimensional. The space $\widehat{M}_{\beta+s}$ is spanned by the set

$$\bigcup_{k \in \Gamma_\mu} \psi(e_{s-k}, M_{\beta+k}).$$

Choose a basis $\{\delta_1, \ldots, \delta_n\}$ of the lattice Γ_μ and introduce a norm on Γ_μ:

$$\|k\| = \sum_{j=1}^{n} |k_j|, \quad \text{where } k = \sum_{j=1}^{n} k_j \delta_j.$$

By Corollary 3.9 there exists $m \in \mathbb{N}$ such that $\Omega_{p,q}^{(m,h)}$ belongs to the annihilator of M for all $p, q, h \in \Gamma_\mu$.

We claim that $\widehat{M}_{\beta+s}$ is equal to

$$\mathrm{Span}\left\{\psi(e_{s-k}, M_{\beta+k}) \;\middle|\; \|k\| \leq \frac{nm}{2}\right\}. \tag{4.5}$$

To prove this claim we need to show that for all $p \in \Gamma_\mu$ and for all $u \in M_{\beta+p}$, $\psi(e_{s-p}, u)$ belongs to (4.5). We prove this by induction on $\|p\|$. If $|p_j| \leq \frac{m}{2}$ for all $j = 1, \ldots, n$, the claim holds trivially. Suppose that for some j we have $|p_j| > \frac{m}{2}$. Let us assume $p_j < -\frac{m}{2}$, the case $p_j > \frac{m}{2}$ is treated in the same way. We have that $p + \delta_j, p + 2\delta_j, \ldots, p + m\delta_j$ all have norms less than $\|p\|$. Since e_0 acts on $M_{\beta+p}$ with a non-zero scalar, we can write $u = e_0 v$ for some $v \in M_{\beta+p}$. Let us verify that

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \psi(e_{s-p-i\delta_j}, e_{i\delta_j} v) = 0$$

in $\widehat{M}$. Indeed,

$$\begin{aligned}\sum_{i=0}^{m} (-1)^i \binom{m}{i} \psi(e_{s-p-i\delta_j}, e_{i\delta_j} v)(t^r) &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} e_{s+\mu\cdot r-p-i\delta_j} e_{i\delta_j} v \\ &= \Omega^{(m,\delta_j)}_{s+\mu\cdot r-p,0} v = 0.\end{aligned}$$

Thus

$$\psi(e_{s-p}, u) = -\sum_{i=1}^{m} (-1)^i \binom{m}{i} \psi(e_{s-p-i\delta_j}, e_{i\delta_j} v). \tag{4.6}$$

By induction assumption the right hand side of (4.6) belongs to (4.5), and so does $\psi(e_{s-p}, u)$, which proves the claim. Hence

$$\widehat{M}_{\beta+s} = \mathrm{Span}\left\{\psi(e_{s-k}, M_{\beta+k}) \;\middle|\; \|k\| \leq \frac{nm}{2}\right\}$$

and it is finite-dimensional. The theorem is proved. $\square$

Using this result for the solenoidal Lie algebras we can easily establish its analogue for the Lie algebra of vector fields on a torus.

Theorem 4.10. *Let M be a cuspidal module for the Lie algebra W_n of vector fields on a torus. Then its A-cover $\widehat{M}$ is cuspidal.*

Proof. Fix a basis $\{\mu_1, \ldots, \mu_n\}$ of $\mathbb{C}^n$ such that every μ_j is generic. Then as a vector space we can decompose W_n into a direct sum of its subalgebras:

$$W_n = W_{\mu_1} \oplus \ldots \oplus W_{\mu_n}.$$

The module M when restricted to each solenoidal Lie algebra W_{μ_j} remains cuspidal. Let $\widehat{M}$ be the A-cover of the W_n-module M. The weight space $\widehat{M}_\lambda$ is spanned by the set

$$\left\{\psi(t^k d_{\mu_j}, M_{\lambda-k}) \;\middle|\; k \in \mathbb{Z}^n,\, j = 1, \ldots, n\right\}.$$

By Theorem 4.9 a subspace spanned by

$$\left\{\psi(t^k d_{\mu_j}, M_{\lambda-k}) \;\middle|\; k \in \mathbb{Z}^n\right\}$$

is finite-dimensional for each $j = 1, \ldots, n$. Thus $\widehat{M}_\lambda$ is finite-dimensional as well. Since $\widehat{M}$ is an A-module, it is cuspidal. $\square$

As a corollary of Theorems 4.9 and 4.10 we obtain Theorem 4.1 by applying Proposition 4.5 (3). We conjecture that in fact the condition $M = \mathcal{L}M$ in Theorem 4.1 is superfluous.

We have thus established a close relation between cuspidal W_n-modules and cuspidal AW_n-modules. The general structure of cuspidal AW_n-modules is well-understood. These modules are parametrized by finite-dimensional representations of a certain infinite-dimensional Lie algebra [2]. Let us recall that construction. Let $\mathcal{J} = \mathrm{Der}\, \mathbb{C}[t_1, \dots, t_n]$, which is the *jet* Lie algebra of W_n (see [3] for the definition of a jet of a Lie algebra). The Lie algebra $\mathcal{J}$ has a natural $\mathbb{Z}$-grading

$$\mathcal{J} = \mathop{\oplus}_{k=-1}^{\infty} \mathcal{J}_k,$$

where $\mathcal{J}_{-1}$ is spanned by $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$ and $\mathcal{J}_0 \cong gl_n$. Set

$$\mathcal{J}_+ = \mathop{\oplus}_{k=0}^{\infty} \mathcal{J}_k.$$

If (V, ρ) is a finite-dimensional representation of $\mathcal{J}_+$ then

$$\rho(\mathcal{J}_k) = 0 \quad \text{for } k \gg 0. \tag{4.7}$$

Theorem 4.11. *([2]) Let M be a cuspidal AW_n-module with the set of weights $\beta + \mathbb{Z}^n$ for some $\beta \in \mathbb{C}^n$. Then there exists a finite-dimensional representation (V, ρ) of $\mathcal{J}_+$ such that*

$$M \cong t^\beta \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \otimes V$$

with the natural action of A and the following W_n-action:

$$(t^m d_j)(t^s \otimes v) = s_j t^{s+m} \otimes v + \sum_{k \in \mathbb{Z}_+^n \setminus \{0\}} \frac{m^k}{k!} t^{s+m} \otimes \rho\left(t^k \frac{\partial}{\partial t_j}\right) v, \tag{4.8}$$

where $m \in \mathbb{Z}^n$, $s \in \beta + \mathbb{Z}^n$, $m^k = m_1^{k_1} \dots m_n^{k_n}$, $k! = k_1! \dots k_n!$ and $v \in V$.

In particular, one can easily obtain from Theorem 4.11 a classification of simple cuspidal AW_n-modules ([9], [2]), which we will present in the next section.

Remark 4.12. Note that (4.8) together with (4.7) implies that a cuspidal AW_n-module always has polynomial structure constants, i.e., it is a module with a polynomial action.

We conclude this section with the discussion of duality in the category of cuspidal modules.

Define the graded dual of a weight module M as

$$M^* = \mathop{\oplus}_{\lambda} (M_\lambda)^*.$$

It is clear that M^* is again a weight module.

Lemma 4.13. *If M is an $A\mathcal{L}$-module then its graded dual M^* is also an $A\mathcal{L}$-module.*

Verification of this lemma is straightforward and we omit its proof.

Proposition 4.14. *Let M be a weight $\mathcal{L}$-module. Then the map*

$$\pi^* : M \to \left(\widehat{M^*}\right)^*$$

given by

$$\pi^*(u)(\psi(x,\xi)) = \xi(xu), \quad \textit{for } u \in M,\ x \in \mathcal{L},\ \xi \in M^*,$$

is a homomorphism of $\mathcal{L}$-modules with kernel $\{u \in M \mid \mathcal{L}u = 0\}$.

Proof. Let us show that π^* is a homomorphism of $\mathcal{L}$-modules. We need to show that for $y \in \mathcal{L}$ we have $y\pi^*(u) = \pi^*(yu)$. Indeed,

$$(y\pi^*(u))(\psi(x,\xi)) = -\pi^*(u)(y\psi(x,\xi)) = -\pi^*(u)(\psi([y,x],\xi) - \pi^*(u)(\psi(x,y\xi))$$
$$= -\xi([y,x]u) - (y\xi)(xu) = -\xi([y,x]u) + \xi(y(xu)) = \xi(x(yu)) = \pi^*(yu)(\psi(x,\xi)).$$

The statement about the kernel of π^* is obvious. □

Corollary 4.15. *Let M be a cuspidal $\mathcal{L}$-module with no 1-dimensional submodules. Then M can be embedded into a cuspidal AW_n-module.*

5. Classification of simple W_n-modules with finite-dimensional weight spaces

In this section we establish the classification of simple W_n-modules with finite-dimensional weight spaces which was originally conjectured by Eswara Rao [9]. First we are going to present two classes of modules: cuspidal simple modules and simple modules of the highest weight type. Then we shall prove that these two classes give a complete list.

Cuspidal simple modules. These modules have a geometric origin – they are modules of tensor fields on a torus ([22], [11], [23]).

Let U be a finite-dimensional simple gl_n-module and let $\beta \in \mathbb{C}^n$. The module of tensor fields $T(U,\beta)$ is the space

$$T(U,\beta) = t^\beta \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \otimes U$$

with a W_n-action given as follows:

$$(t^m d_a)(t^s \otimes u) = s_a t^{s+m} \otimes u + \sum_{p=1}^{n} m_p t^{s+m} \otimes E_{pa} u,$$

where $m \in \mathbb{Z}^n$, $s \in \beta + \mathbb{Z}^n$, $u \in U$ and E_{pa} is an $n \times n$ matrix with 1 in position (p,a) and zeros elsewhere. It is easy to check that the modules of tensor fields $T(U,\beta)$ are AW_n-modules.

Let V be the natural n-dimensional representation of gl_n and let $\Lambda^k V$ be its k-th exterior power, $k = 0, \ldots, n$. Note that $\Lambda^k V$ is a simple gl_n-module for all $k = 0, \ldots, n$. The modules $\Omega^k(\beta) = T(\Lambda^k V, \beta)$ are the modules of differential k-forms. These modules form the de Rham complex

$$\Omega^0(\beta) \xrightarrow{d} \Omega^1(\beta) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(\beta).$$

The differential d of the de Rham complex is a homomorphism of W_n-modules (however it is not a homomorphism of A-modules). Thus the kernels and the images of d are W_n-submodules in $\Omega^k(\beta)$. As a result, for $1 \le k \le n-1$, $\Omega^k(\beta)$ are reducible W_n-modules, while $\Omega^0(\beta)$ and $\Omega^n(\beta)$ are reducible if and only if $\beta \in \mathbb{Z}^n$.

The following Theorem was proved by Eswara Rao (there is also a similar result of Rudakov [22] for the Lie algebra of vector fields on an affine space):

Theorem 5.1. *([8], see also [13]) Let U be a finite-dimensional simple gl_n-module and let $\beta \in \mathbb{C}^n$.*

(1) The module of tensor fields $T(U, \beta)$ is a simple W_n-module unless it is a module $\Omega^k(\beta)$ of differential k-forms, $0 \leq k \leq n$.

(2) The module $\Omega^k(\beta)$, $0 \leq k \leq n$, has a unique simple quotient, which is $d\Omega^k(\beta)$ for $0 \leq k \leq n-1$. The module $\Omega^n(\beta)$ is simple when $\beta \notin \mathbb{Z}^n$ and has a trivial 1-dimensional module as a simple quotient when $\beta \in \mathbb{Z}^n$.

Eswara Rao also classified simple cuspidal AW_n-modules [9]. Cuspidal AW_n-modules without the assumption on simplicity are described in [2].

Theorem 5.2. *([9]) Simple cuspidal AW_n-modules are precisely the modules of tensor fields $T(U, \beta)$ with simple finite-dimensional gl_n-modules U and $\beta \in \mathbb{C}^n$.*

Simple W_n-modules of the highest weight type. Consider a $\mathbb{Z}$-grading on W_n by eigenvalues of $d_n = t_n \frac{\partial}{\partial t_n}$. Write the decomposition of W_n:

$$W_n = W_n^- \oplus W_n^0 \oplus W_n^+$$

into the subalgebras of negative, zero and positive degree. The subalgebra W_n^0 is a semidirect product of W_{n-1} with an abelian ideal:

$$W_n^0 = \operatorname{Der} \mathbb{C}[t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}] \ltimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}] d_n.$$

Let $\widetilde{X}$ be a weight module for W_n^0 with a unique simple quotient X (we could have started with X, but keeping in mind the construction of simple modules $d\Omega^k(\beta)$, we prefer to work in this slightly more general setting). We define simple modules of the highest weight type using the technique of the generalized Verma modules.

Postulate $W_n^+ \widetilde{X} = 0$ and construct the generalized Verma module by inducing:

$$M(\widetilde{X}) = \operatorname{Ind}_{W_n^0 \oplus W_n^+}^{W_n} \widetilde{X} \cong U(W_n^-) \otimes \widetilde{X}.$$

The generalized Verma module $M(\widetilde{X})$ has a unique simple quotient, which we denote $L(\widetilde{X})$. Note that $L(\widetilde{X}) \cong L(X)$.

The group $GL_n(\mathbb{Z})$ acts on the algebra A of Laurent polynomials, which is the group algebra of $\mathbb{Z}^n$. This induces the action of $GL_n(\mathbb{Z})$ by automorphisms on $W_n = \operatorname{Der} A$. For each $g \in GL_n(\mathbb{Z})$ we can consider a twisted module $L(\widetilde{X})^g$, where the action of W_n is composed with an automorphism g.

We call the modules $L(\widetilde{X})^g$ simple W_n-modules of the *highest weight type*. Whereas the generalized Verma module $M(\widetilde{X})$ always contains infinite-dimensional weight spaces, the situation may improve with passing to its simple quotient $L(\widetilde{X})$.

Theorem 5.3. *([1], see also [5]) Let $\widetilde{X}$ be a W_n^0-module with a polynomial action (see [1] for the definition). Then the simple W_n-module of the highest weight type $L(\widetilde{X})$ has finite-dimensional weight spaces.*

In particular, we can take a simple finite-dimensional gl_{n-1}-module U, and let $\widetilde{X}$ be a W_{n-1}-module of tensor fields $T(U, \beta)$ for some $\beta \in \mathbb{C}^{n-1}$. We fix $\gamma \in \mathbb{C}$ and define the action of the abelian ideal as follows:

$$(t^m d_n)(t^s \otimes u) = \gamma t^{s+m} \otimes u, \quad \text{for } m \in \mathbb{Z}^{n-1},\ s \in \beta + \mathbb{Z}^{n-1},\ u \in U.$$

The resulting module $\widetilde{X} = T(U, \beta, \gamma)$ is a W_n^0-module with a polynomial action [5] and has a unique simple quotient. Thus the corresponding simple W_n-module of the highest weight type $L(\widetilde{X}) = L(U, \beta, \gamma)$ has finite-dimensional weight spaces.

The structure of the simple modules $L(U, \beta, \gamma)$ was studied in [4], where their vertex operator realizations are given.

Now we can state the main classification result presented in Introduction.

Theorem 5.4. *(1) Every simple W_n-module with finite-dimensional weight spaces is either cuspidal or of the highest weight type.*

(2) A simple cuspidal W_n-module is isomorphic to one of the following:

- *a module of tensor fields $T(U, \beta)$, where $\beta \in \mathbb{C}^n$ and U is a finite-dimensional simple gl_n-module different from the k-th exterior power of the natural n-dimensional gl_n-module, $0 \leq k \leq n$;*
- *a submodule $d\Omega^k(\beta) \subset \Omega^{k+1}(\beta)$, $0 \leq k < n$;*
- *a trivial 1-dimensional W_n-module.*

(3) A module of the highest weight type is isomorphic to $L(U, \beta, \gamma)^g$ for some finite-dimensional simple gl_{n-1}-module U, $\beta \in \mathbb{C}^{n-1}$, $\gamma \in \mathbb{C}$ and $g \in GL_n(\mathbb{Z})$.

Proof. Part (1) of the Theorem was proved by Mazorchuk-Zhao ([21], Theorem 1). More precisely, they prove that a simple W_n-module with finite-dimensional weight spaces is either cuspidal or is isomorphic to a module of the highest weight type $L(X)^g$ for some $g \in GL_n(\mathbb{Z})$ and some simple cuspidal W_n^0-module X. Let us analyze these two cases.

Case of cuspidal modules. The classification in this case follows from the following lemma:

Lemma 5.5. *Let M be a simple cuspidal W_n-module. Then M is a simple quotient of a module of tensor fields $T(U, \beta)$ for some simple gl_n-module U and $\beta \in \mathbb{C}^n$.*

Proof of Lemma. The claim holds for the trivial 1-dimensional module since it is the simple quotient of the module $\Omega^n(0)$. Let us now assume that M is a non-trivial simple module. Then $W_n M = M$ and there is a surjective homomorphism π from a cuspidal AW_n-module $\widehat{M}$ to M by Proposition 4.5(3) and Theorem 4.10.

Consider a composition series of AW_n-submodules in $\widehat{M}$:

$$0 = \widehat{M}_0 \subset \widehat{M}_1 \subset \ldots \subset \widehat{M}_{\ell-1} \subset \widehat{M}_\ell = \widehat{M}$$

with the quotients $\widehat{M}_i / \widehat{M}_{i-1}$ being simple AW_n-modules. Let s be the smallest integer such that $\pi(\widehat{M}_s) \neq 0$. Since M is a simple W_n-module we have $\pi(\widehat{M}_s) = M$ and $\pi(\widehat{M}_{s-1}) = 0$. This gives us a surjective homomorphism of W_n-modules

$$\bar{\pi} : \widehat{M}_s / \widehat{M}_{s-1} \to M.$$

By Theorem 5.2, $\widehat{M}_s / \widehat{M}_{s-1}$ is isomorphic to a module of tensor fields $T(U, \beta)$ for some simple gl_n-module U and $\beta \in \mathbb{C}^n$. This completes the proof of the lemma. □

The classification of simple cuspidal W_n-modules now follows from Lemma 5.5 and Theorem 5.1.

Case of simple modules of the highest weight type. Let M be a simple module $L(X)^g$ for some $g \in GL_n(\mathbb{Z})$ and some simple cuspidal W_n^0-module X. Guo-Liu-Zhao [12] have reduced the description of these modules to the classification of simple cuspidal W_{n-1}-modules. More precisely, they proved ([12], Theorem 3.3) that either $X \cong T(U, \beta, \gamma)$ for some finite-dimensional simple gl_{n-1}-module U, $\beta \in \mathbb{C}^{n-1}$ and some non-zero $\gamma \in \mathbb{C}$, or the abelian ideal $\mathbb{C}[t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}] d_n$ acts trivially on X. The latter case is the only one that we need to consider. In

this setting X is just a simple cuspidal W_{n-1}-module. By Lemma 5.5, X is a simple quotient of a module $\widetilde{X} = T(U, \beta)$ for some finite-dimensional simple gl_{n-1}-module U and $\beta \in \mathbb{C}^{n-1}$. Then as a W_n^0-module, $\widetilde{X}$ is isomorphic to $T(U, \beta, 0)$ and $M \cong L(U, \beta, 0)^g$. This completes the proof of the theorem. □

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